Convergence rate for the numerical solution of SDEs with distributional drift in Besov spaces

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Introduction

- The current work aims to create a numerical scheme to approximate the solution to an SDE of the form

$$X_t = x + \int_0^t b\left(s, X_s\right) ds + W_t, \tag{1}$$

for $t \in [0,T]$, where W_t is a Brownian motion and $b(t,\cdot) \in \mathcal{C}^{-\beta}(\mathbb{R})$ with $0 < \beta < 1/2$, where $\mathcal{C}^{-\beta}(\mathbb{R})$ is the Hölder-Zygmund space, which is a special case of Besov space.

Assumptions

The following assumptions hold all throuought this poster:

1. For $\beta \in (0, 1/4)$ the coefficient $b \in C_T\mathcal{C}^{-\beta}$.
2. There exists $(b^N)_{N\geq 0}$ such that $\lim_{N\to\infty} b^N = b$ in $C_T\mathcal{C}^{-\beta}$.

Euler scheme and rate of convergence

Main result

- Let us have the following Euler scheme as in [2]

$$X^{Nm}_{t_{k+1}} = X_{t_k} + b^N\left(t_k, X^{Nm}_{t_k}\right)\left(t_{k+1} - t_k\right) + \left(W_{t_{k+1}} - W_{t_k}\right). \tag{2}$$

- Note that this is not a standard Euler scheme since we require to control two parameters, namely the amount of time steps m and the element of the sequence N, simultaneously.

Theorem: Let X_t be the solution to the SDE (1) and X^{Nm}_t be the Euler approximation of the solution with $m > 0$ time steps, then it holds that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left|X^{Nm}_t - X_t\right|\right] \leq c m^{-\frac{1}{2}+\mu+\epsilon} \tag{3}$$

where

$$\mu = \frac{1}{2}\cdot\frac{\beta}{(1/2-\beta)(1-2\beta)+\beta} \tag{4}$$

for any $\epsilon > 0$.

Regularised SDE and its convergence and convergence rate

Step 1

- With $(b^N)_{N\geq 0}$ we can define equations for all $t \in [0,T]$ and see that the solution converges to X as proved in [3]

$$X^N_t = x + \int_0^t b^N(s, X^N_s) ds + W_t. \tag{5}$$

- For any $\alpha > \beta$ we have

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left|X^N_t - X_t\right|\right] \leq C_\alpha \|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}, \tag{6}$$

as $N \to \infty$, where

$$C_\alpha = 2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T^2 e^{5\lambda t}. \tag{7}$$

- Where u is the solution to the Kolmogorov equation associated to (1)

Convergence of the Euler scheme to the regularised SDE

Step 2

- The following result is borrowed from [2].
- Consider the Euler scheme from (2), then if Assumption 1 and 2 holds and

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|X^{Nm}_t - X^N_t|\right] \leq A(N)m^{-1} + B(N)m^{-\frac{1}{2}} \tag{8}$$

where

$$A_N = c\left\|b^N\right\|_{L^\infty}\left(1 + \left\|\nabla b^N\right\|_{L^\infty}\right)$$
$$B_N = c'\left(\left\|\nabla b^N\right\|_{L^\infty} + \left[b^N\right]_{\frac{1}{2},L^\infty}\right), \tag{9}$$

with $c, c' > 0$ constants independent of (N, m).

Combination of the two convergence results

Step 3

The two results above can be combined to get the main result in the following way:

- Triangle inequality with the bounds of $\mathbb{E}[|X^N - X|]$ and $\mathbb{E}[|X^{Nm} - X^N|]$.
- Bound the L^∞-norms that appear in A_N, B_N, and also the difference $\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}$.
- Use the selected N and rearrange the terms.

Examples of b and b^N

Example of b

- We can use a $b = \frac{\partial}{\partial x} f(x)$ for $f \in \mathcal{C}^{-\beta+1}_c$.
- In particular, we can chose $f(x) = \psi B^H(x,\omega)$, for some $\omega \in \Omega$ and a cut-off function $\psi \in C^\infty$ with compact support.

Example of b^N

- An example of a drift b^N is proposed in [5] where it is mentioned that for $b \in C_T\mathcal{C}^{-\beta'}$ for $\beta' < \beta$ we can define the sequence $(b^N)_{N\geq 0}$ as

$$b^N(t,\cdot) := \phi_N * b(t,\cdot) \tag{10}$$

where $\phi_N := p_{f(N)}$, p is the heat kernel, and $f(N) \to 0$ as $N \to \infty$

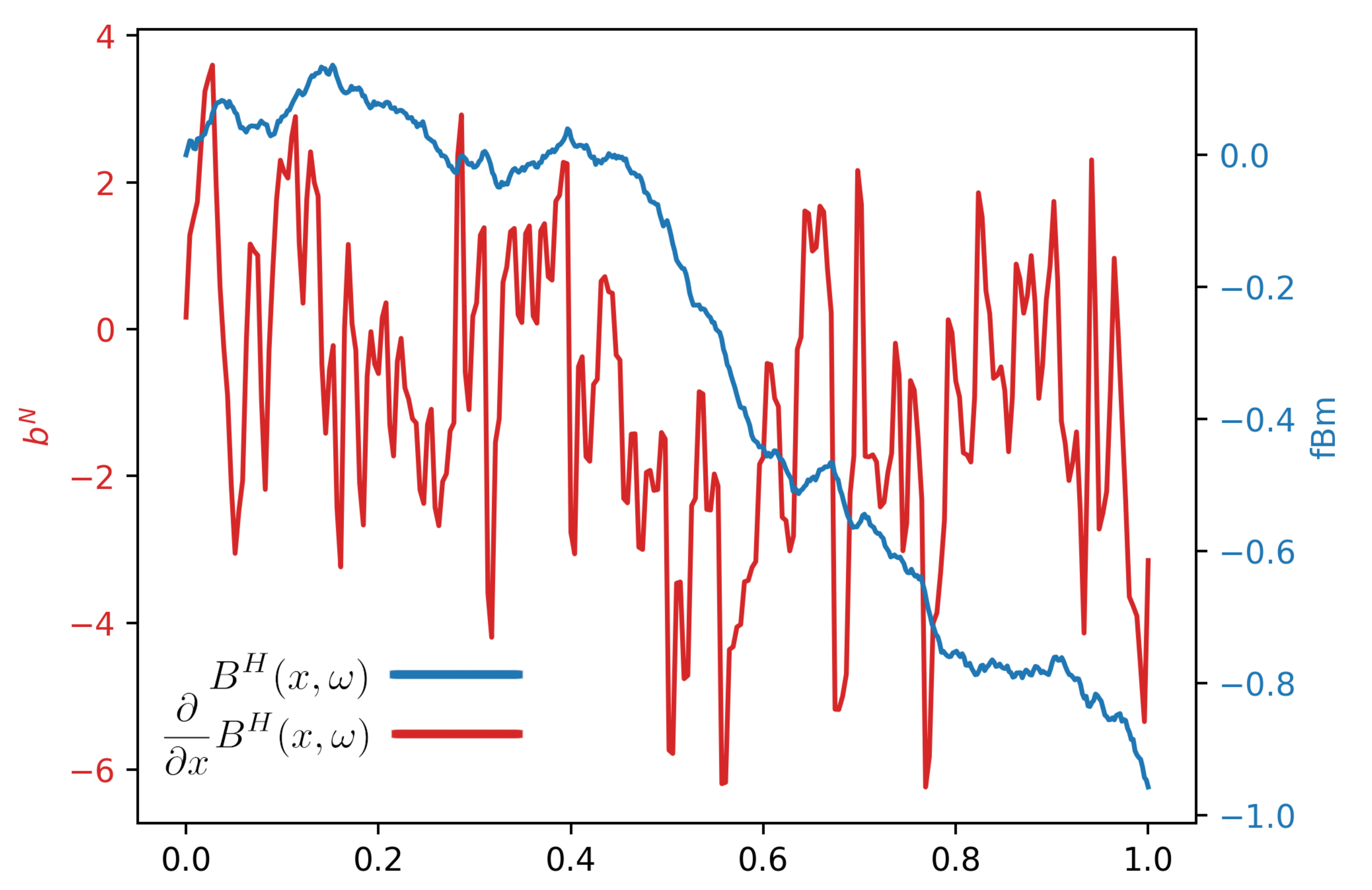

Fig. 1: Numerical example of $b^N \in C_T\mathcal{C}^{-0.15}$. The coefficient b^N is $\frac{\partial}{\partial x}B^H$, and the B^H is the path of fBm from which it was generated.

Special cases $\beta \approx 0, \beta \approx 1/4$

Note that in the convergence rate (3) we will have

$$\beta \approx 0 \implies m^{-1/2-\epsilon} \qquad \beta \approx 1/4 \implies m^{-1/6-\epsilon}$$

Related work

- In [3] the theoretical framework to study SDEs with distributional drift in spaces $H^{-\beta}_{\tilde{q}q}$ was developed and in [2] a numerical scheme for such equations is studied.
- SDEs with drift in Besov spaces is analysed in [6] formulated as the martingale problem for distributional drifts.
- In [1] it is found a convergence rate of $\frac{1}{2}$ for the case with measurable drifts (think $\beta \approx 0$).
- An optimal convergence rate was found in [4] when the drift lives in a Besov space and the SDE is driven by a fractional Brownian motion with $H < 1/2$. In contrast our result uses Brownian motion, i.e. $H = 1/2$.

Future work

- With the foundations presented, the aim is to obtain a numerical scheme for a McKean SDE of the form studied in [5]

$$X_t = X_0 + \int_0^t F(v(s, X_s))b(s, X_s)ds + W_t \tag{11}$$

where $v(t,\cdot)$ is the law density of X_t, $F : \mathbb{R} \to \mathbb{R}$ is non-linear, and b is a distribution as before.

Here one of the further challenges is in approximating numerically the function $v(t,x)$, which is the solution to the Fokker-Planck equation.

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