Euler scheme for SDEs with distributional drifts

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Joint work with Elena Issoglio and Jan Palczewski

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Outline

The main question

- For an SDE

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0, \end{cases}$$

- We say that a discrete time approximation X^m, where m are the time steps, converges strongly with order $\gamma > 0$ at time T if

$$\mathbb{E}|X_T - X_T^m| \leq cm^{-\gamma}.$$

- Classical result tells us that if μ and σ are Lipschitz continuous and have sub-linear growth we have $\gamma = 1/2$, and if $\mu, \sigma \in \mathbb{R}$, we have $\gamma = \infty$. [26, *Kloeden & Platen*].
- How far can we take the Euler scheme?

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Main aims

- Convergence rate of the Euler-Maruyama (EM) scheme for SDEs with distributional coefficients.
- Convergence rate of the Euler-Maruyama (EM) scheme for McKean-Vlasov SDEs with distributional coefficients.
- Implement said numerical methods and compare the empirical and theoretical rate.

Reference[1]

- **Preprint:** C. J., Issoglio, Palczewski. *Convergence rate of numerical scheme for SDEs with a distributional drift in Besov space*. arXiv:2309.11396 [8]
- **Implementation:** C. J., Issoglio, Palczewski. *Implementation of the Numerical Methods*. doi.org/10.5281/zenodo.8239606 [7]

Preprint

Implementation

[1] The work on McKean equations is in the last stages of preparation.

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Setting

We study the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t \\ X_0 = x_0, \end{cases}$$

- where the drift $b \in C^{1/2}([0, T]; C^{-\beta}(\mathbb{R}))$ for $0 < \beta < 1/2$,
- and W_t is a one-dimensional Brownian motion and $x_0 \in \mathbb{R}$.

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- [17, *Flandoli, Issoglio & Russo (2017)*] d-dimension with $b(t,\cdot) \in H^{-s}_{q,\tilde{q}}$, virtual solutions.
- [24, 22, *Issoglio & Russo (2023 & 2024)*] d-dimension with $b(t,\cdot) \in C^{(-\beta)+}$, virtual solutions and martingale problem equivalence.
- Related works include [32, *Veretennikov (1981)*], [2, *Bass & Chen (2001)*], [18, 19, *Flandoli, Russo & Wolf (2003 & 2004)*] [6, *Cannizzaro & Chouk (2018)*] [11, *Chaudru de Raynal & Menozzi (2022)*].

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Theoretical results

- Existence of solutions is formulated through virtual solutions, i.e.

$$X_t = x+v(0,x)-v(t,X_t)+\lambda\int_0^t v(s,X_s)ds+\int_0^t (v_x(s,X_s)+1)\,dW_s,$$

- where v is the solution to

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- We then define an auxiliary process $Y_t = v(t, X_t) + X_t$.
  - Denote $\phi(t, x) = u(t, x) + x$.
- The function ϕ has a space inverse ψ, and we see that $X_t = \psi(\cdot, Y_t)$.
- Prove that ψ is 2-Lipschitz and then we can use

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  - Approximate the solution to the SDE X_t with a regularised SDE X_t^N.
  - Create a numerical approximation $X_t^{N,m}$ of the regularised SDE.
- Convergence rate $r(\beta) = \frac{\left(\frac{1}{2}-\beta\right)^2}{2\left(\frac{1}{2}-\beta\right)^2+\beta+1}$.
  - $\lim_{\beta\downarrow 0} r(\beta) = \frac{1}{6}$.
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The McKean-Vlasov SDE (MVSDE) which concerns us is

$$\begin{cases} dX_t = F(v(t, X_t))b(t, X_t)dt + dW_t \\ X_0 \sim \mu. \end{cases}$$

- Where F is a non-linear function,
- $v(t, \cdot)$ is the law density of X_t,
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- Instead of using propagation of chaos results.
- It is proved by [23, *Issoglio & Russo*] that in this case X_t has a law density which is the weak solution of the Fokker-Planck PDE

$$\begin{cases} \partial_t \rho = \frac{1}{2}\partial_{xx}\rho - \partial_x(\rho F(\rho) b) \\ \rho(0) = \rho_0 \end{cases}$$

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- The PDE is one dimensional so we opt to solve it numerically.

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- We prove that the convergence rate for this equation is the same as for the linear case, i.e.
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Numerical implementation

- The trick of our implementation is in the choice of b.
- We select the drift to be $b = \partial_x B \in C^{-\beta}$ for some function $B \in C^{1-\beta}$.
- The smoothed drift is $b^N = p_{\frac{1}{N}} * b = p_{\frac{1}{N}} * \partial_x B = \partial_x p_{\frac{1}{N}} * B$.

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- Our choice of the function B is a path of a fBm B^H where $0 < H < 1$.
- The advantages of using an fBm are twofold:
  1. Having a function that is rough in a whole interval.
  2. The regularity properties of fBm allow us to identify the space where the drift lives.
- For the McKean SDE we have to additionally find out ρ^N.
- Since the PDE is one-dimensional we can get away with solving it directly.

Numerical implementation

- Our choice of the function B is a path of a fBm B^H where $0 < H < 1$.
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  1. Having a function that is rough in a whole interval.
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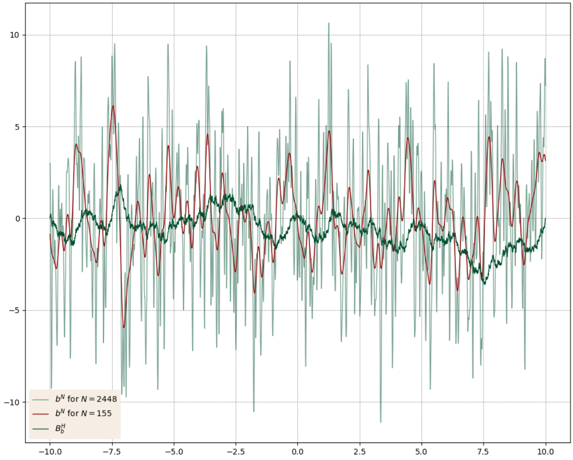
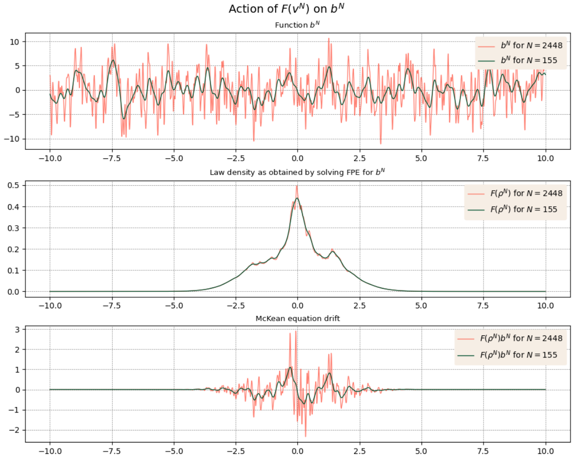


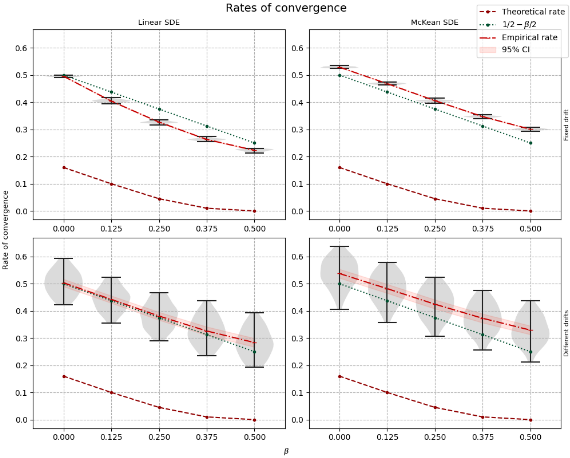
Drift b^N compared to fBm with Hurst coefficient $H = 1/2 - \epsilon$.

For the McKean-Vlasov SDE



Function b^N compared to fBm with Hurst coefficient $H = 1/2 - \epsilon$ and $F(x) = \sin(x)$ displaying the effect of the law of the process.

Theoretical, empirical and hypothetical rates



Comparison of the empirical convergence rate for McKean-Vlasov SDEs with respect to different $\hat{\beta}$ and with hypothesis rate extending [13, *Dareiotis, Gerencsér & Lê*].

Conclusion and future work

- We get a bound for the convergence rate of SDEs with distributional drifts driven by a standard Brownian motion
- The convergence rate obtained is at least $1/6$ close to the measurable case and 0 for the limit case of the regularity.
- Explore the usage of the stochastic sewing lemma as done by [20] to get the optimal rate of convergence.

Thank you!

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