Euler scheme for SDEs with distributional drifts

Luis Mario Chaparro Jáquez

Joint work with Elena Issoglio and Jan Palczewski

School of Mathematics, University of Leeds

The Changing (of) times

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Outline

- For an SDE

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0, \end{cases}$$

- We say that a discrete time approximation X^m, where m are the time steps, converges strongly with order $\gamma > 0$ at time T if

$$\mathbb{E}|X_T - X_T^m| \leq cm^{-\gamma}.$$

- Classical result tells us that if μ and σ are Lipschitz continuous and have sub-linear growth we have $\gamma = 1/2$, and if $\mu, \sigma \in \mathbb{R}$, we have $\gamma = \infty$. [24, *Kloeden & Platen*].
- How much can we break the regularity of μ and σ and have convergent numerical methods?

The main question

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Main aims

- Convergence rate of the Euler-Maruyama (EM) scheme for SDEs with distributional coefficients.
- Convergence rate of the Euler-Maruyama (EM) scheme for McKean-Vlasov SDEs with distributional coefficients.
- Implement said numerical methods and compare the empirical and theoretical rate.

Reference

- C. J., Issoglio, Palczewski. *Convergence rate of numerical scheme for SDEs with a distributional drift in Besov space*. 2024. arXiv:2309.11396
- C. J., Issoglio, Palczewski. *Euler-Maruyama scheme for McKean-Vlasov equations with distributional drift*. 2025+.

We study the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t \\ X_0 = x_0, \end{cases}$$

- where the drift $b \in C^{1/2}([0, T]; C^{-\beta}(\mathbb{R}))$ for $0 < \beta < 1/2$,
- and W_t is a one-dimensional Brownian motion and $x_0 \in \mathbb{R}$.

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Literature: Theoretical analysis

- [15, *Flandoli, Issoglio & Russo (2017)*] d-dimension with $b(t,\cdot) \in H^{-s}_{q,\tilde{q}}$, virtual solutions.
- [22, 20, *Issoglio & Russo (2023 & 2024)*] d-dimension with $b(t,\cdot) \in C^{(-\beta)+}$, virtual solutions and martingale problem equivalence.
- Related works include [30, *Veretennikov (1981)*], [2, *Bass & Chen (2001)*], [16, 17, *Flandoli, Russo & Wolf (2003 & 2004)*] [6, *Cannizzaro & Chouk (2018)*] [9, *Chaudru de Raynal & Menozzi (2022)*].

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Literature: Numerical methods

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- [18, *Goudenège, Haress & Richard (2022)*] $b \in \mathcal{B}^{\gamma}_{p}$, noise fBm with $0 < H < 1/2$, stochastic sewing lemma.
- [11, *Dareiotis, Gerencsér & Lê (2023)*] find a convergence rate $(1+\alpha)/2$ for SDEs with drifts with regularity of order α.
- Many other related works [3, *Bossy & Diop (2015)*], [28, *Neuenkirch & Szölgyenyi (2021)*], [23, *Jourdain & Menozzi (2021)*], [5, *Butkovski, Dareiotis & Gerencsér (2021)*], [26, *Lê & Ling (2022)*].

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Theoretical results

- Existence of solutions is formulated through virtual solutions, i.e.

$$X_t = x + v(0,x) - v(t,X_t) + \lambda \int_0^t v(s,X_s)ds + \int_0^t (v_x(s,X_s)+1)\,dW_s,$$

- where v is the solution to

$$\begin{cases} v_t + \frac{1}{2}v_{xx} + bv_x = \lambda v - b \\ v(T) = 0. \end{cases}$$

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- We then define an auxiliary process $Y_t = v(t, X_t) + X_t$.
  - Denote $\phi(t, x) = u(t, x) + x$.
- The function ϕ has a space inverse ψ, and we see that $X_t = \psi(\cdot, Y_t)$.
- Prove that ψ is 2-Lipschitz and then we can use

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- Two step approach:
  - Approximate the solution to the SDE X_t with a regularised SDE X_t^N with drift $b^N = p_{1/N} * b$.
  - Create a numerical approximation $X_t^{N,m}$ of the regularised SDE.
- Convergence rate $r(\beta) = \frac{\left(\frac{1}{2}-\beta\right)^2}{2\left(\frac{1}{2}-\beta\right)^2+\beta+1}$.
  - $\lim_{\beta\downarrow 0} r(\beta) = \frac{1}{6}$.
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Setting

The McKean-Vlasov SDE (MVSDE) which concerns us is

$$\begin{cases} dX_t = F(v(t, X_t))b(t, X_t)dt + dW_t \\ X_0 \sim \mu. \end{cases}$$

- Where F is a non-linear function,
- $v(t, \cdot)$ is the law density of X_t,
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- Here we use an approach similar to the one for the linear case.
- Instead of using propagation of chaos results.
- It is proved by [21, *Issoglio & Russo*] that in this case X_t has a law density which is the weak solution of the Fokker-Planck PDE

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- Provided we can solve this, we can use similar techniques as for the linear case.
- The PDE is one dimensional so we opt to solve it numerically.

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Numerical implementation

- The trick of our implementation is in the choice of b.
- We select the drift to be $b = \partial_x B \in C^{-\beta}$ for some function $B \in C^{1-\beta}$.
- The smoothed drift is $b^N = p_{\frac{1}{N}} * b = p_{\frac{1}{N}} * \partial_x B = \partial_x p_{\frac{1}{N}} * B$.

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Numerical implementation

- Our choice of the function B is a path of a fBm B^H where $0 < H < 1$.
- The advantages of using an fBm are twofold:
  1. Having a function that is rough in a whole interval.
  2. The regularity properties of fBm allow us to identify the space where the drift lives.
- For the McKean SDE we have to additionally find out ρ^N.
- Since the PDE is one-dimensional we can get away with solving it directly.

Numerical implementation

- Our choice of the function B is a path of a fBm B^H where $0 < H < 1$.
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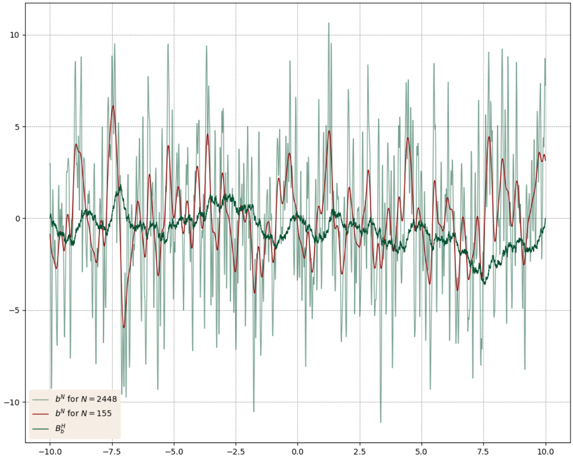
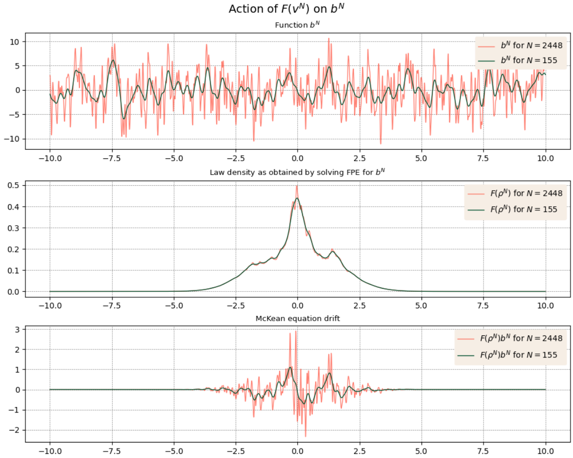
For the linear SDE



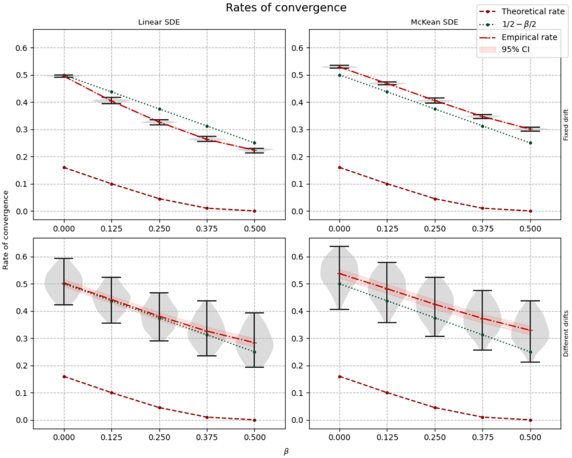
Drift b^N compared to fBm with Hurst coefficient $H = 1/2 - \epsilon$.

For the McKean-Vlasov SDE



Function b^N compared to fBm with Hurst coefficient $H = 1/2 - \epsilon$ and $F(x) = \sin(x)$ displaying the effect of the law of the process.

Theoretical, empirical and hypothetical rates



Comparison of the empirical convergence rate for McKean-Vlasov SDEs with respect to different $\hat{\beta}$ and with hypothesis rate extending [11, *Dareiotis, Gerencsér & Lê*].

Conclusion and future work

- We get a bound for the convergence rate of SDEs with distributional drifts driven by a standard Brownian motion
- The convergence rate obtained is at least $1/6$ close to the measurable case and 0 for the limit case of the regularity.
- Explore the usage of the stochastic sewing lemma as done by [18] to get the optimal rate of convergence.

Thank you!

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