PROBABILITY BASICS

$(\Omega, \mathcal{F}, \mathbb{P})$

$\Omega \to$ SS, $\mathcal{F} \to$ SS, $\mathbb{P} \to$ PROB MEAS

EXAMPLE: DICE

$\Omega = \{1, \dots, 6\}$

$\mathcal{F} = 2^{\Omega}$

$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$ FOR $A \in \mathcal{F}$

$\mathbb{P}(\Omega) = 1$

CONDITIONAL PROB.

$\mathbb{Q}_B(A) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ FOR $\mathbb{P}(B) \neq 0$

$(B, \mathcal{F}_B, \mathbb{Q})$

$\mathbb{P}(A|\Omega) = \mathbb{P}(A)$

RVs (CONTINUOUS)

$X : \Omega \to \mathbb{R}$ ON $(\mathbb{R}, \mathcal{B})$

PROB. DISTRIBUTION

$\mathbb{P}_X(B) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B \in \mathcal{B}\})$

DIST. FUNCTION

$F_X(x) = \mathbb{P}_X((-\infty, x))$ ~~ESTIMATES~~ OF X

DENSITY FUNC

$p(x)$ S.T.

$F_X(x) = \int_{-\infty}^{x} p(y)\, dy$

FOR $\{X_k\}_{k=1,\dots,n}$

$F_{X_1,\dots,X_n} = \mathbb{P}(\{\omega \in \Omega : X_i(\omega) \le x_i, i = 1, \dots, n\})$

MARKOV CHAINS

DEF (MARKOV CHAIN)

$\mathcal{S} \to$ STATE SPACE, $\lambda = (\lambda_i) \to$ INIT DIST

$p_{ij} = p(x_i, x_j) = \mathbb{P}(X_{k+1} = x_j \mid X_k = x_i)$

FOR ALL $k \in T \rightsquigarrow$ TRANSITION PROB.

$p_{ij} \ge 0$ & $\sum_j p_{ij} = 0$ FOR ALL i

X_n IS A T.H.M.C. IF

$\mathbb{P}(X_0 = x_i) = \lambda_i$

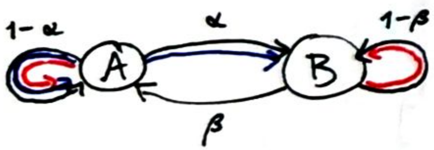
AND

$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_0 = x_0)$

$= \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$

$= p_{n, n+1}$

EXAMPLE OF THMC



1ST STEP

2ND STEP

WHAT ABOUT THE PROBABILITY OF STARTING AT A AND BEING ON A AFTER TWO JUMPS ?

$p_{AA}(2) = \underbrace{(1-\alpha)^2}_{\text{1ST OPT}} + \underbrace{\alpha\beta}_{\text{2ND OPT}}$

$P = \begin{pmatrix} p_{AA} & p_{AB} \\ p_{BA} & p_{BB} \end{pmatrix}$ $\quad P^2_{(AA)} = p_{AA}(2)$

IN GRAL.

$p_{ij}(n) \rightsquigarrow$ n-STEP TRANS. PROB.

AND

$P^n \rightsquigarrow$ n-TRANSITION MATRIX

TIME HOMOGENEITY GIVES YOU THIS

THM (CHAPMAN-KOLMOGOROV EQN) (CKE)

$P^{n+m} = P^n P^m$

$P(n+m) = P(n)P(m)$

DENOTE $p(n) \rightsquigarrow$ DIST AT TIME n

$p(n) = p(n-1) P = \lambda \underbrace{P \cdots P}_{n}$

$p(n) = \lambda P^n$

DIFFUSION PROCESSES

DEF (MARK. PROC) $(\mathbb{R}, \mathcal{B})$, $T \subseteq \mathbb{R}$

$(s, x; t, B) \longmapsto P(s, x; t, B)$

FOR $s, t \in T$ S.T. $s \le t$, $x \in \mathbb{R}$, $B \in \mathcal{B}$

1. FIXED s, t, x, $B \mapsto P(s, x; t, B)$ IS A PROB. MEAS. ON $\mathcal{B}$
2. FIXED s, t, B, $x \mapsto P(s, \cdot\,; t, B)$ IS MEASURABLE ON $\mathcal{B}$
3. FOR $s, t, u \in T$ S.T. $s \le t \le u$, ALL $x \in \mathbb{R}$ AND $B \in \mathcal{B}$ (CKE HOLDS)

$P(s, x; u, B) = \int_{\mathbb{R}} P(t, y; u, B)\, P(s, x; t, dy)$

7

$\{X_t\}_{t\in T}$ IS A MARKOV PROC.

IF $\forall\, t,u \in T$, $t \le u$

$$P(t, X_t; u, y) =: P(X_u = y \mid \mathcal{F}_{\le t})$$

$$P(s,x;t,B) = \int_B p(s,x;t,y)\, dy$$

$P(s,x;t,B)$ TRANSITION PROBABILITY

CKE

$$p(s,x;u,z) = \int_{\mathbb{R}} p(t,y;u,z)\, p(s,x;t,y)\, dy$$

8

DEF (DIFF. PROC.)

A M.P. S.T. $\forall \varepsilon > 0$, $s \ge 0$, $x \in \mathbb{R}$ AND WELL DEF. $\mu(x,t)$, $\sigma(x,t)$, THE FOLLOWING HOLDS

(A) $$\lim_{t \downarrow s} \frac{1}{t-s} \int_{|y-x| > \varepsilon} P(s,x;t,dy) = 0$$

(B) $$\lim_{t \downarrow s} \frac{1}{t-s} \int_{|x-y| < \varepsilon} (y-x)\, P(s,x;t,dy) = \mu(x,t)$$

(C) $$\lim_{t \downarrow s} \frac{1}{t-s} \int_{|x-y| < \varepsilon} (y-x)^2\, P(s,x;t,dy) = \sigma^2(x,t)$$

9

EXAMPLE BM

GAUSSIAN PROCESS, INDEP. INCREMENTS

- $P(\{\omega \in \Omega : W_0 = 0\}) = 1$
- $\mathbb{E}\, W_t = 0$
- $\mathrm{Var}(W_t - W_s) = t - s$

$$p_W = \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(y-x)^2}{2(t-s)}}$$

FOR THIS THE LIMITS (A–C) HOLD AND

$\mu_W(x,t) = 0$, $\sigma_W^2(x,t) = 1$

IT'S THE FUNDAM. SOLN.

$$\frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial x^2} = 0 \quad \& \quad \frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}$$

10

IN GRAL FOR ANY DIFF.-PROC

P IS THE SOLN OF

> KOLMOGOROV FORWARD EQN (FOKKER PLANCK)

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial t}[\mu(t,y)p] - \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(t,y)p] = 0$$

S, X FIXED

AND

> KOLMOGOROV BACKWARD EQN

$$\frac{\partial p}{\partial t} + \mu(s,x)\frac{\partial p}{\partial x} - \frac{1}{2}\sigma^2(s,x)\frac{\partial^2 p}{\partial x^2} = 0$$

T, Y FIXED

NOTE THAT

$$\mathcal{L}_u = \mu(u,x)\frac{\partial}{\partial x} - \frac{1}{2}\sigma^2(u,x)\frac{\partial^2}{\partial x^2}$$

11

AND FORMAL ADJOINT

$$\mathcal{L}_u^* = \frac{\partial}{\partial y}[\mu(u,y)] - \frac{\partial^2}{\partial y^2}[\sigma^2(u,y)]$$

PROOFS NOT SEMINAR FRIENDLY

THM (FOKKER-PLANCK)

SUPPOSE (A–C) HOLD LOCALLY UNIFORMLY ON x AND μ, σ^2 ARE LOCALLY BOUNDED, THEN p SATISFIES ☺.

PROOF (VEEERY HANDWAVY)

$f \in C_0^\infty(\mathbb{R})$ $\qquad h = (t-s)$

$$\frac{d}{dt}\int_{\mathbb{R}} f(y)\, P(s,x;t,dy) = \lim_{t\downarrow s}\frac{1}{t-s}\left(\int_{\mathbb{R}} f(y)\, P(s,x;t+h,dy) - \int_{\mathbb{R}} f(z)\, P(s,x;t,dz)\right)$$

12

APPLY CKE TO RHS

$$\lim_{h\to 0} h^{-1} \iint_{\mathbb{R}\mathbb{R}} (f(y) - f(z))\, P(t,z;t+h,dy)\, P(s,x,t,dz)$$

USE (A–C) AND TAYLOR EXP THE INTEG WRT y BECOMES

$$\frac{1}{2}\sigma^2(t,z)\frac{\partial^2 f}{\partial z^2} + \mu(t,z)\frac{\partial f}{\partial z}$$

SINCE CONV AS $h \to 0$ IS UNIF IN z

$$\frac{d}{dt}\int_{\mathbb{R}} f(y)\, P(s,x,t,dy) = \lim_{h\to 0}\int_{\mathbb{R}}\left(\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial z^2} + \mu\frac{\partial f}{\partial z}\right) P(s,x;t,dz)$$