Contents

1 Overview of numerical methods for SDEs

What is an SDE

- Think of an ODE with an extra term of "noise"

$$\frac{dy}{dx} = f(y, x) + \xi \tag{1}$$

- The most common notation is

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned} \tag{2}$$

- Where μ and σ are functions of t, x and W is a Brownian Motion

Why numerical methods?

- Just as in deterministic differential equations, regardless of the type of coefficients we have We don't normally know the explicit solution of an SDE
- Very often one needs to compute specific quantities from a process
- Some examples of such quantities are:
- Functionals $\mathbb{E}[f(X_t)]$ for a explicit given function f in order to solve parabolic PDEs via Monte Carlo approximation TO DO: Add the PDE just in case?
- On Talay 1990 we can find a broader discussion on this topic, in particular there is a number of results about numerical methods to compute different quantities from a process X_t
- Higham 2001 has an introduction to numerical methods with code examples

When can we use them?

- For equations with good regularity properties the Euler-Maruyama (Euler method for short) method has been used for a long time.
- Even higher order methods have been devised, they require stronger conditions by having dependency on the derivatives of the SDE coefficients
- Strong and weak approximations have been studied
  - Euler method is if strong order 0.5 and weak order 1.0.
  - Milstein method is if strong and weak order 1.0.
  - There are methods with strong orders 1.5 and 2.0, and weak orders 2.0, 3.0 and 4.0.

  TO DO: Add strong and weak convergence expectation thing
- An extensive discussion on this topics can be found in the book **Numerical Solution of Stochastic Differential Equations** Kloeden and Platen 1999
- And for a more hands-on approach see **Numerical Solution of SDE Through Computer Experiments** Kloeden, Platen, and Schurz 1997

How are they obtained?

- Numerical schemes are derived by truncating the **Itô-Taylor** expansion of the SDE we wish to approximate.
- The Euler method comes from the first two terms of the Itô-Taylor expansion, this is

$$X_{t_{n+1}} = X_{t_n} + \mu(t_n, X_{t_n})(t_{n+1} - t_n) + \sigma(t_n, X_{t_n})(W_{t_{n+1}} - W_{t_n}) \tag{3}$$

- I use the colours to indicate what each term represents
- In general the Itô-Taylor expansion for (2) reads TO DO: Add the expansion in general for the article version from Kloeden and Platen 1999

$$X_t = \tag{4}$$

How do we know they converge?

- Let us focus on the Euler scheme, which is the most widely studied.
- The Milstein scheme is very also commonly studied because of it's stronger covergence rate.
- However it requires stronger restrictions on the coefficients and it will not necessarily yield better results than the simpler Euler scheme, that's why the later holds its relevance in theory and practice
- The classical case for which a converegence rate of $1/2$ is obtained, require the coefficients from the SDE of interest to be Lipschitz continuous and have linear growth.
- For equation (2) this means

$$\begin{aligned} |\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq C_1|x-y| \\ |\mu(t,x)| + |\sigma(t,x)| &\leq C_2(1+|x|) \\ |\mu(s,x) - \mu(t,y)| + |\sigma(s,x) - \sigma(t,x)| &\leq C_3(1-|x|)|s-t|^{1/2} \end{aligned} \tag{5}$$

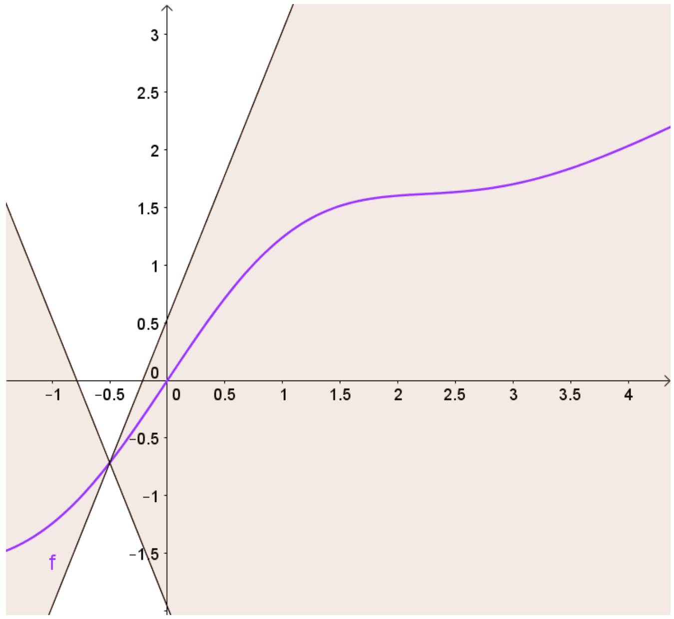

Figure 1: Visualization of the Lipschitz condition

Lipschitz condition visualized

2 Irregular coefficients and where to find them

What do we mean by irregular coefficients?

- The moment we steer away from the regularity conditions previously stated we are talking about irregular coefficients.
- The fact that we have a very strong continuity condition for the coefficients of the SDE gives place to the question:
- **How much can we relax this condition and still obtain useful results?**
- What about finitely many discontinuity points?
- What about having a Hölder condition on an interval?

Two examples from finance

- **The use of the Black-Scholes models as a CAPM**
- CAPM = Capital Asset Pricing Model
- People tend to update the model on a regular basis to account for the constant changes on the volatility (diffusion coefficient σ on (2))
- Practicioners could benefit of having some model with a step function of time as diffusion, which requires relaxing the Lipschitz condition to allow discontinuities

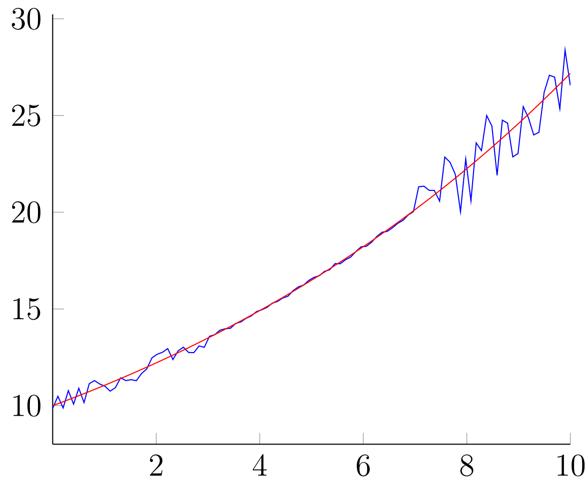

Figure 2: Example of diffusion process with volatility dependent on time

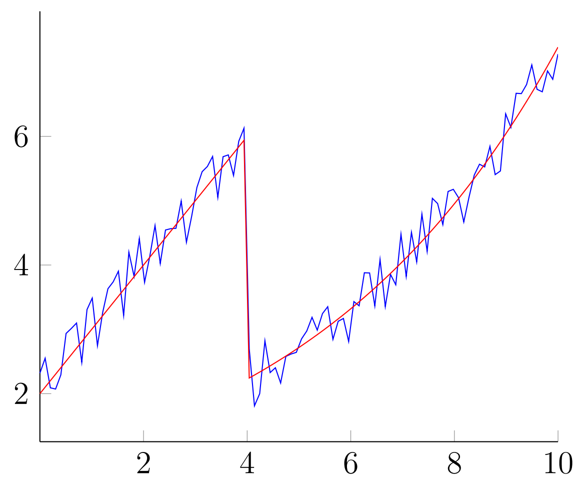

Figure 3: Example of diffusion process with piecewise defined drift

- **A company takeover**
- If we have models as SDEs for two companies and one absorbs the other, then then we might see a sudden jump in their prices can occur
- This can be explained with a discontinuity on the drift

3 Irregular coefficients in the literature

Measurable coefficients in SDEs

- In recent years there has been an sparkling interest on relaxing the conditions under which we can find solutions to SDEs

- Veretennikov 1981 finds that for the case in which Equation (2) has $\sigma = 1$ (the unit matrix on the multidimensional case) and μ is merely bounded, there is a strong solution
- Gyongy and Martinez 2001 find a regularisation by noise result for SDEs with locally unbounded drifts

Distributional coefficients in SDEs

- Cannizzaro and Chouk 2018 frame it as a martingale problem and approach it with paracontrolled distributions
- Issoglio and Russo 2022 have an equation with a drift in a negative Besov space and a unit diffusion, they frame it as a martingale problem and introduce a notion of solution which agrees with their martingale problem solution
- Related works include Delarue and Diel 2016 and Chaudru de Raynal and Menozzi 2022

4 Numerical schemes for irregular coefficients

What problems do we face?

- In principle, nothing stops us from just implement the function we want on the computer
- The main problem is the lack of theory which assures us a numerical method will converge
- In practice indeed it's sometimes possible to observe behaviours similar to the Euler scheme with classical conditions
- But only a few cases have been studied in the numerical sense by obtaining results on a rate of convergence or numerical stability
- Furthermore, there are case that are more challenging and require some clever tricks to represent even the coefficeints that are used
- Any numerical method requires finitely many points, so in a point of discontinuity we will potentially have a bigger difference between points than the numerical method can handle
- TO DO: The example below perhaps won't work, see Peano's example
- Take the deterministic initial value problem

  $$y' = f, \;\; y(0) = 0 \tag{6}$$

  where

  $$f(x) = \begin{cases} 1 & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

  f is clearly discontinuous at -1 and 1. If we use the Euler scheme

  $$y_{n+1} = y_n + \Delta x f(y_n) \tag{8}$$

  TO DO: double check this example and finish, find a counterexample that works

SDEs with Measurable coefficients

- There were two important approaches regarding drifts which are still functions in the classical sense μ:
- **Allow discontinuities on a small set**
- And outside of this set the usual Lipschitz condition is assumed to hold
- Under this assumptions, the Euler scheme has convergence rate 1/2 and the Milstein scheme has 3/4 Müller-Gronbach and Yaroslavtseva 2019
- **Having a mild condition**
- Such as α-Hölder continuity for α arbitrarily small
- And it was suggested that the rate of convergence deteriorates as α becomes smaller Menoukeu Pamen ar
- Dareiotis and Gerencsér 2020 found that just by having measurable and bounded coefficients we can get a convergence rate 1/2 just as for the regular case.

Distributional coefficients in SDEs

- Regularisation by noise allows us to use the noise to regularise even distributional drifts
- De Angelis, Germain, and Issoglio 2022 work with drifts in fractional Sobolev spaces and propose the usage of Haar and Faber basis to create numerical approximation which will be further mollified by applying the heat semigroup
- This is the convolution with the standard Gaussian density function
- Goudenège, Haress, and Richard 2022 study equations with fractional Brownian motion with a Hurst parameter less than a half ($H < 1/2$) thus benefiting from a rougher noise as a noise drifts in Hölder-Zygmund spaces
- Find convergence rate which deteriorates as the regularity of the drift drops, they also implement particular examples of drifts

On my current work

- I am working with the SDE that is studied by Issoglio and Russo 2022

$$dX_t = b(t, X_t)dt + W_t \tag{9}$$

- In particular with the drift $b =$ "$\frac{d}{dx}$"B^H where B^H is a fractional Brownian motion There are a number of reason, both theoretical and numerical to use the fBm as a drift for this equation:
  - Another example of function which could be used is x^α which blows up at zero and therefore doesn't have a derivative
  - FBM on the other hand is badly behaved everywhere

  - It also happens that it is α-Hölder continuous for any $\alpha < H$ where H is its Hurst parameter

- We compute a mollified "derivative" of a Hölder continuous function which we generate numerically
- The computation of a numerical derivative poses its own challenges when the function is not smooth
- Normally this is achieved using finite differences
- This method exploits the definition of derivative

$$\frac{f(x+h) - f(x)}{h} \tag{10}$$

- So that when the derivative we want to compute the the deivative of a funtion which has jumps, the derivative doesn't exist as a function and things go wrong
- If I have several examples of fBm mention regularity (Hölder continuity) instead of spaces
- Be clear that the final object we work with (the drift) is not the fBm but its derivative
- How do we address this situation is not trivial to implement
- The approximation of the drift for the numerical approximations is defined as the convolution of b with the Gaussian density p_{f_m} where $f_m = 1/m^\eta$ is the variance
- TO DO: Mention more of the challenges: The variance is vanishing and the convolution at the limit is the convolution with the heat kernel
- TO DO: How do we define this vanishing variance? has to be linked with the Euler scheme (step size)

Picture of fBm

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0 \le t \le T} \mathbb{E}[|X_t^m - X_t|] \le c m^{-r(\hat{\beta})+\epsilon}, \tag{11}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2} - \hat{\beta}\right)^2}{2\left(\frac{1}{2} - \hat{\beta}\right)^2 + \hat{\beta} + 1} \tag{12}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1-\beta$ This basically is the Hölder continuity of the drift

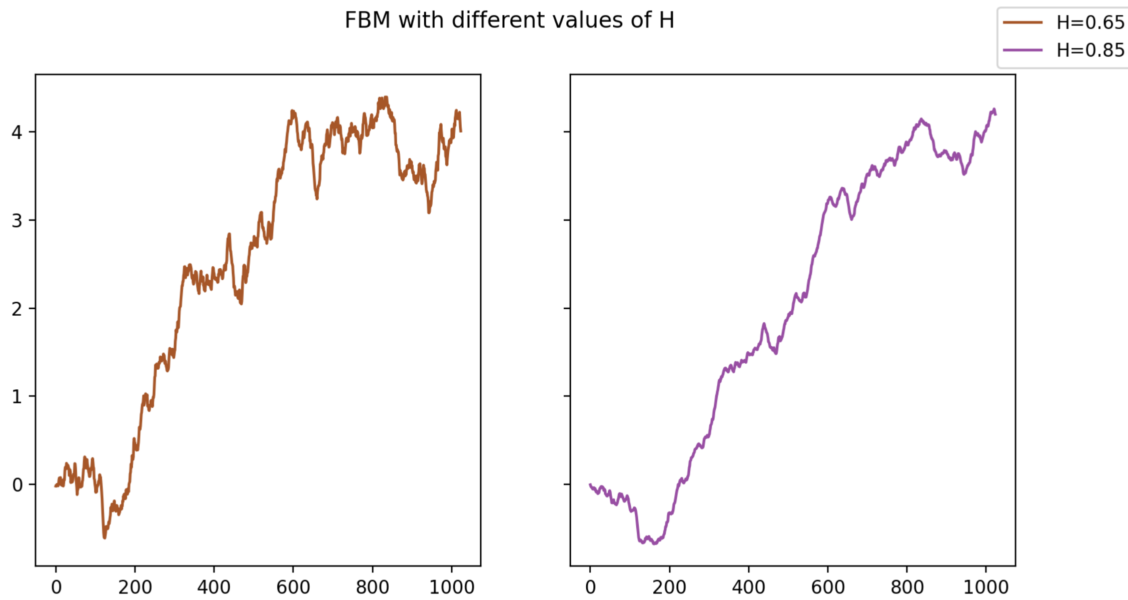

Figure 4: This picture illustrates two examples of fBm. The object we need as a drift is the weak derivative of such paths

- And it looks scary, but our limit cases are more revelaling
- On the limit cases:
  - $\beta \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\beta \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

And about the implementation...

- One step towards the implementation comes from the fact that we want a mollified version of this derivative
- And we can compute $b \star p_{f_m} = \frac{d}{dx} B^H \star p_{f_m}$ as $B^H \star \frac{d}{dx} p_{f_m}$
- This is still not particularly straighforward to implement
- Some of the issues are:
  - The errors between consecutive Euler approximations don't seem to decrease as fast as they should
  - Implementing the drift can be challenging because of the vanishing variance

Conclusion

- Irregular coefficients on SDEs have gotten a lot of attention not only for theoretical reasons but because of their potential practical uses

- Theoretical results are useful but I believe it's necessary to give much more attention to the intrincacies of the implementations
- My implementation will potentially help generalise the numerical methods for one particular type of SDE

Buon appetito!

5 Potential questions

- Strong and weak solutions, strong and weak convergence Are this things related? Can you only study strong/weak convergence for strong/weak solutions?

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