Challenges in numerical methods for SDEs with irregular coefficients

Luis Mario Chaparro Jáquez

University of Leeds
School of Mathematics

June 19th 2023

1 Overview of numerical methods for SDEs

2 Irregular coefficients and where to find them

3 Irregular coefficients in the literature

4 Numerical schemes for irregular coefficients

Table of Contents

1 Overview of numerical methods for SDEs

2 Irregular coefficients and where to find them

3 Irregular coefficients in the literature

4 Numerical schemes for irregular coefficients

What is an SDE

- Think of an ODE with an extra term of “noise”

$$\frac{dy}{dx} = f(y, x) + \xi \quad (1)$$

- The most common notation is

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned} \quad (2)$$

- Where μ and σ are functions of t, x and W is a Brownian Motion

What is an SDE

- Think of an ODE with an extra term of "noise"

$$\frac{dy}{dx} = f(y, x) + \xi \tag{1}$$

- The most common notation is

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned} \tag{2}$$

- Where μ and σ are functions of t, x and W is a Brownian Motion

What is an SDE

- Think of an ODE with an extra term of "noise"

$$\frac{dy}{dx} = f(y, x) + \xi \tag{1}$$

- The most common notation is

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0 \end{aligned} \tag{2}$$

- Where μ and σ are functions of t, x and W is a Brownian Motion

Why numerical methods?

- We don't normally know the explicit solution of an SDE
- Very often one needs to compute specific quantities from a process
- On Talay 1990 we can find a broader discussion on this topic, in particular there is a number of results about numerical methods to compute different quantities from a process X_t
- Higham 2001 has an introduction to numerical methods with code examples

Why numerical methods?

- We don't normally know the explicit solution of an SDE
- Very often one needs to compute specific quantities from a process
- On Talay 1990 we can find a broader discussion on this topic, in particular there is a number of results about numerical methods to compute different quantities from a process X_t
- Higham 2001 has an introduction to numerical methods with code examples

Why numerical methods?

- We don't normally know the explicit solution of an SDE
- Very often one needs to compute specific quantities from a process
- On Talay 1990 we can find a broader discussion on this topic, in particular there is a number of results about numerical methods to compute different quantities from a process X_t
- Higham 2001 has an introduction to numerical methods with code examples

Why numerical methods?

- We don't normally know the explicit solution of an SDE
- Very often one needs to compute specific quantities from a process
- On Talay 1990 we can find a broader discussion on this topic, in particular there is a number of results about numerical methods to compute different quantities from a process X_t
- Higham 2001 has an introduction to numerical methods with code examples

How are they obtained?

- Numerical schemes are derived by truncating the **Itô-Taylor** expansion of the SDE we wish to approximate.
- The Euler method comes from the first two terms of the Itô-Taylor expansion, this is

$$X_{t_{n+1}} = X_{t_n} + \mu(t_n, X_{t_n})(t_{n+1} - t_n) + \sigma(t_n, X_{t_n})(W_{t_{n+1}} - W_{t_n}) \tag{3}$$

How are they obtained?

- Numerical schemes are derived by truncating the **Itô-Taylor** expansion of the SDE we wish to approximate.
- The Euler method comes from the first two terms of the Itô-Taylor expansion, this is

$$X_{t_{n+1}} = X_{t_n} + \mu(t_n, X_{t_n})(t_{n+1} - t_n) + \sigma(t_n, X_{t_n})(W_{t_{n+1}} - W_{t_n}) \tag{3}$$

When can we use them?

- For equations with good regularity properties the Euler-Maruyama method has been used for a long time.
- Strong and weak approximations have been studied

$$\mathbb{E}[|X^m - X|] \leq C_1 m^{-r_s} \quad \mathbb{E}[|X^m| - |X|] \leq C_2 m^{-r_w} \qquad (4)$$

- A good reference resource on the topic is the book **Numerical Solution of Stochastic Differential Equations** Kloeden and Platen 1999
- And for a more hands-on approach see **Numerical Solution of SDE Through Computer Experiments** Kloeden, Platen, and Schurz 1997

When can we use them?

- For equations with good regularity properties the Euler-Maruyama method has been used for a long time.
- Strong and weak approximations have been studied

$$\mathbb{E}[|X^m - X|] \leq C_1 m^{-r_s} \quad \mathbb{E}[|X^m| - |X|] \leq C_2 m^{-r_w} \tag{4}$$

- A good reference resource on the topic is the book **Numerical Solution of Stochastic Differential Equations** Kloeden and Platen 1999
- And for a more hands-on approach see **Numerical Solution of SDE Through Computer Experiments** Kloeden, Platen, and Schurz 1997

When can we use them?

- For equations with good regularity properties the Euler-Maruyama method has been used for a long time.
- Strong and weak approximations have been studied

$$\mathbb{E}[|X^m - X|] \leq C_1 m^{-r_s} \quad \mathbb{E}[|X^m| - |X|] \leq C_2 m^{-r_w} \tag{4}$$

- A good reference resource on the topic is the book **Numerical Solution of Stochastic Differential Equations** Kloeden and Platen 1999
- And for a more hands-on approach see **Numerical Solution of SDE Through Computer Experiments** Kloeden, Platen, and Schurz 1997

When can we use them?

- For equations with good regularity properties the Euler-Maruyama method has been used for a long time.
- Strong and weak approximations have been studied

$$\mathbb{E}[|X^m - X|] \leq C_1 m^{-r_s} \quad \mathbb{E}[|X^m| - |X|] \leq C_2 m^{-r_w} \tag{4}$$

- A good reference resource on the topic is the book **Numerical Solution of Stochastic Differential Equations** Kloeden and Platen 1999
- And for a more hands-on approach see **Numerical Solution of SDE Through Computer Experiments** Kloeden, Platen, and Schurz 1997

How do we know they converge?

- Let us focus on the Euler scheme, which is the most widely studied.
- The classical case for which a converegence rate of 1/2 is obtained, require the coefficients from the SDE of interest to be Lipschitz continuous and have linear growth.
- For equation (2) this means

$$
\begin{aligned}
|\mu(t,x)-\mu(t,y)|+|\sigma(t,x)-\sigma(t,y)| &\leq C_1|x-y| \\
|\mu(t,x)|+|\sigma(t,x)| &\leq C_2(1+|x|) \\
|\mu(s,x)-\mu(t,y)|+|\sigma(s,x)-\sigma(t,x)| \leq C_3(1-|x|)|s-t|^{1/2}
\end{aligned}
\tag{5}
$$

How do we know they converge?

- Let us focus on the Euler scheme, which is the most widely studied.
- The classical case for which a converegence rate of 1/2 is obtained, require the coefficients from the SDE of interest to be Lipschitz continuous and have linear growth.
- For equation (2) this means

$$|\mu(t,x)-\mu(t,y)|+|\sigma(t,x)-\sigma(t,y)|\leq C_1|x-y|$$
$$|\mu(t,x)|+|\sigma(t,x)|\leq C_2(1+|x|)$$
$$|\mu(s,x)-\mu(t,y)|+|\sigma(s,x)-\sigma(t,x)|\leq C_3(1-|x|)|s-t|^{1/2} \tag{5}$$

How do we know they converge?

- Let us focus on the Euler scheme, which is the most widely studied.
- The classical case for which a converegence rate of 1/2 is obtained, require the coefficients from the SDE of interest to be Lipschitz continuous and have linear growth.
- For equation (2) this means

$$
\begin{aligned}
|\mu(t,x)-\mu(t,y)|+|\sigma(t,x)-\sigma(t,y)| &\leq C_1|x-y| \\
|\mu(t,x)|+|\sigma(t,x)| &\leq C_2(1+|x|) \\
|\mu(s,x)-\mu(t,y)|+|\sigma(s,x)-\sigma(t,x)| &\leq C_3(1-|x|)|s-t|^{1/2}
\end{aligned} \tag{5}
$$

Lipschitz condition visualized

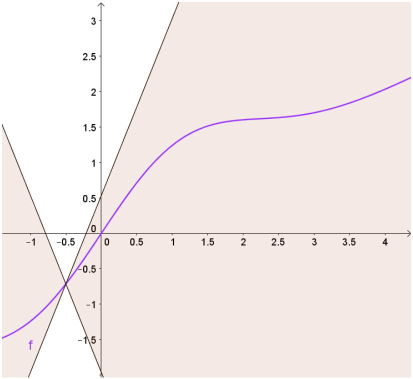

Figure: Visualization of the Lipschitz condition

Table of Contents

1 Overview of numerical methods for SDEs

2 Irregular coefficients and where to find them

3 Irregular coefficients in the literature

4 Numerical schemes for irregular coefficients

What do we mean by irregular coefficients?

- The moment we steer away from the regularity conditions previously stated we are talking about irregular coefficients.
- How much can we relax this condition and still obtain useful results?

What do we mean by irregular coefficients?

- The moment we steer away from the regularity conditions previously stated we are talking about irregular coefficients.
- **How much can we relax this condition and still obtain useful results?**

Two examples from finance

- **The use of the Black-Scholes models as a CAPM**
- **A company takeover**

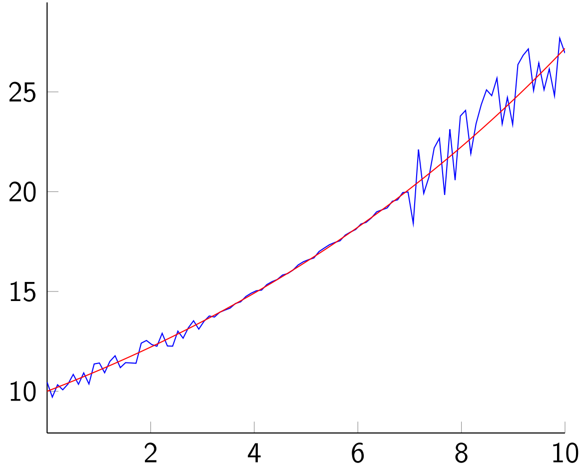

Figure: Example of diffusion process with volatility dependent on time

Two examples from finance

- The use of the Black-Scholes models as a CAPM
- A company takeover

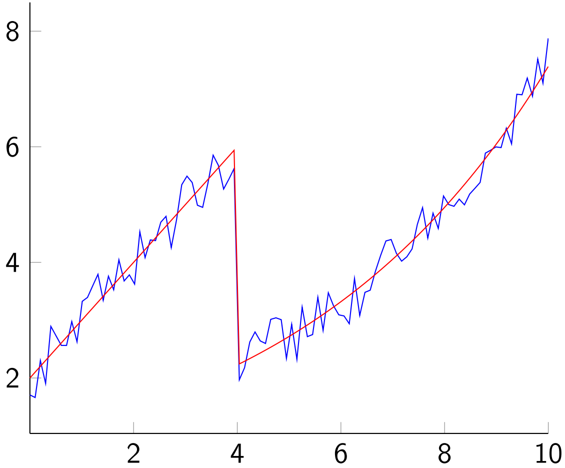

Figure: Example of diffusion process with piecewise defined drift

Table of Contents

1 Overview of numerical methods for SDEs

2 Irregular coefficients and where to find them

3 Irregular coefficients in the literature

4 Numerical schemes for irregular coefficients

Measurable coefficients in SDEs

- Veretennikov 1981 finds that for the case in which Equation (2) has $\sigma = 1$ and μ is merely bounded, there is a strong solution
- Gyongy and Martinez 2001 find a regularisation by noise result for SDEs with locally unbounded drifts

Measurable coefficients in SDEs

- Veretennikov 1981 finds that for the case in which Equation (2) has $\sigma = 1$ and μ is merely bounded, there is a strong solution
- Gyongy and Martinez 2001 find a regularisation by noise result for SDEs with locally unbounded drifts

Distributional coefficients in SDEs

- Cannizzaro and Chouk 2018 frame it as a martingale problem and approach it with paracontrolled distributions
- Issoglio and Russo 2022 have an equation with a drift in a negative Besov space and a unit diffusion, they frame it as a martingale problem and introduce a notion of solution which agrees with their martingale problem solution
- Related works include Delarue and Diel 2016 and Chaudru de Raynal and Menozzi 2022

Distributional coefficients in SDEs

- Cannizzaro and Chouk 2018 frame it as a martingale problem and approach it with paracontrolled distributions
- Issoglio and Russo 2022 have an equation with a drift in a negative Besov space and a unit diffusion, they frame it as a martingale problem and introduce a notion of solution which agrees with their martingale problem solution
- Related works include Delarue and Diel 2016 and Chaudru de Raynal and Menozzi 2022

Distributional coefficients in SDEs

- Cannizzaro and Chouk 2018 frame it as a martingale problem and approach it with paracontrolled distributions
- Issoglio and Russo 2022 have an equation with a drift in a negative Besov space and a unit diffusion, they frame it as a martingale problem and introduce a notion of solution which agrees with their martingale problem solution
- Related works include Delarue and Diel 2016 and Chaudru de Raynal and Menozzi 2022

Table of Contents

1 Overview of numerical methods for SDEs

2 Irregular coefficients and where to find them

3 Irregular coefficients in the literature

4 Numerical schemes for irregular coefficients

What problems do we face?

- In principle, nothing stops us from just implement the function we want on the computer
- The main problem is the lack of theory which assures us a numerical method will converge

What problems do we face?

- In principle, nothing stops us from just implement the function we want on the computer
- The main problem is the lack of theory which assures us a numerical method will converge

SDEs with Measurable coefficients

- There were two important approaches regarding drifts which are still functions in the classical sense μ:
- Allow discontinuities on a small set
- Having a mild condition
- Dareiotis and Gerencsér 2020 found that just by having measurable and bounded coefficients we can get a convergence rate 1/2 just as for the regular case.

SDEs with Measurable coefficients

- There were two important approaches regarding drifts which are still functions in the classical sense μ:
- **Allow discontinuities on a small set**
- **Having a mild condition**
- Dareiotis and Gerencsér 2020 found that just by having measurable and bounded coefficients we can get a convergence rate 1/2 just as for the regular case.

SDEs with Measurable coefficients

- There were two important approaches regarding drifts which are still functions in the classical sense μ:
- **Allow discontinuities on a small set**
- **Having a mild condition**
- Dareiotis and Gerencsér 2020 found that just by having measurable and bounded coefficients we can get a convergence rate 1/2 just as for the regular case.

SDEs with Measurable coefficients

- There were two important approaches regarding drifts which are still functions in the classical sense μ:
- **Allow discontinuities on a small set**
- **Having a mild condition**
- Dareiotis and Gerencsér 2020 found that just by having measurable and bounded coefficients we can get a convergence rate 1/2 just as for the regular case.

Distributional coefficients in SDEs

- Regularisation by noise allows us to use the noise to regularise even distributional drifts
- De Angelis, Germain, and Issoglio 2022 work with drifts in fractional Sobolev spaces and propose the usage of Haar and Faber basis to create numerical approximation which will be further mollified by applying the heat semigroup
- Goudenège, Haress, and Richard 2022 study equations with fractional Brownian motion $(H < 1/2)$ as a noise drifts in Hölder-Zygmund spaces

Distributional coefficients in SDEs

- Regularisation by noise allows us to use the noise to regularise even distributional drifts
- De Angelis, Germain, and Issoglio 2022 work with drifts in fractional Sobolev spaces and propose the usage of Haar and Faber basis to create numerical approximation which will be further mollified by applying the heat semigroup
- Goudenège, Haress, and Richard 2022 study equations with fractional Brownian motion $(H < 1/2)$ as a noise drifts in Hölder-Zygmund spaces

Distributional coefficients in SDEs

- Regularisation by noise allows us to use the noise to regularise even distributional drifts
- De Angelis, Germain, and Issoglio 2022 work with drifts in fractional Sobolev spaces and propose the usage of Haar and Faber basis to create numerical approximation which will be further mollified by applying the heat semigroup
- Goudenège, Haress, and Richard 2022 study equations with fractional Brownian motion $(H < 1/2)$ as a noise drifts in Hölder-Zygmund spaces

On my current work

- I am working with the SDE that is studied by Issoglio and Russo 2022

$$dX_t = b(t, X_t)dt + W_t \tag{6}$$

- In particular with the drift $b = \text{``}\frac{d}{dx}\text{''} B^H$ where B^H is a trajectory of fractional Brownian motion
- The approximation of the drift for the numerical approximations is defined as the convolution of b with the Gaussian density p_{f_m} where $f_m = 1/m^\eta$ is the variance

On my current work

- I am working with the SDE that is studied by Issoglio and Russo 2022
$$dX_t = b(t, X_t)dt + W_t \tag{6}$$
- In particular with the drift $b = \text{``}\frac{d}{dx}\text{''} B^H$ where B^H is a trajectory of fractional Brownian motion
- The approximation of the drift for the numerical approximations is defined as the convolution of b with the Gaussian density p_{f_m} where $f_m = 1/m^\eta$ is the variance

On my current work

- I am working with the SDE that is studied by Issoglio and Russo 2022

$$dX_t = b(t, X_t)dt + W_t \tag{6}$$

- In particular with the drift $b = \text{``}\frac{d}{dx}\text{''} B^H$ where B^H is a trajectory of fractional Brownian motion
- The approximation of the drift for the numerical approximations is defined as the convolution of b with the Gaussian density p_{f_m} where $f_m = 1/m^\eta$ is the variance

Picture of fBm

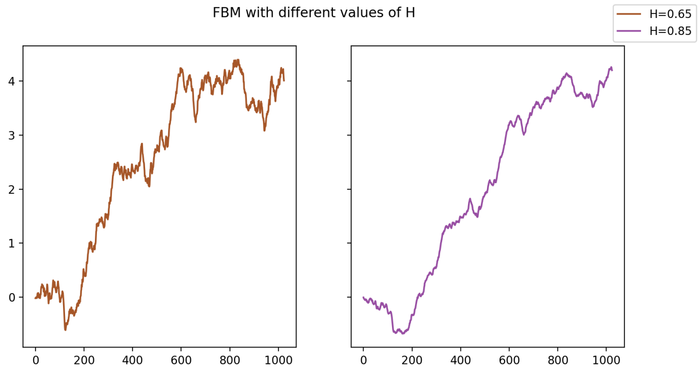

Figure: This picture illustrates two examples of fBm. The object we need as a drift is the weak derivative of such paths

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t^m - X_t|] \leq cm^{-r(\hat{\beta})+\epsilon}, \tag{7}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2} - \hat{\beta}\right)^2}{2\left(\frac{1}{2} - \hat{\beta}\right)^2 + \hat{\beta} + 1} \tag{8}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1 - \hat{\beta}$
- On the limit cases:
  - $\beta \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\beta \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0 \le t \le T} \mathbb{E}[|X_t^m - X_t|] \le c m^{-r(\hat{\beta})+\epsilon}, \tag{7}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2} - \hat{\beta}\right)^2}{2\left(\frac{1}{2} - \hat{\beta}\right)^2 + \hat{\beta} + 1} \tag{8}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1 - \hat{\beta}$
- On the limit cases:
  - $\beta \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\beta \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t^m - X_t|] \leq cm^{-r(\hat{\beta})+\epsilon}, \tag{7}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2} - \hat{\beta}\right)^2}{2\left(\frac{1}{2} - \hat{\beta}\right)^2 + \hat{\beta} + 1} \tag{8}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1 - \hat{\beta}$
- On the limit cases:
  - $\hat{\beta} \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\hat{\beta} \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t^m - X_t|] \leq cm^{-r(\hat{\beta})+\epsilon}, \tag{7}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2}-\hat{\beta}\right)^2}{2\left(\frac{1}{2}-\hat{\beta}\right)^2+\hat{\beta}+1} \tag{8}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1 - \hat{\beta}$
- On the limit cases:
  - $\beta \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\beta \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t^m - X_t|] \leq cm^{-r(\hat{\beta})+\epsilon}, \tag{7}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2} - \hat{\beta}\right)^2}{2\left(\frac{1}{2} - \hat{\beta}\right)^2 + \hat{\beta} + 1} \tag{8}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1 - \hat{\beta}$
- On the limit cases:
  - $\beta \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\beta \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

Convergence of the Euler scheme

- For this equation we found a strong convergence rate

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X_t^m - X_t|] \leq c m^{-r(\hat{\beta})+\epsilon}, \tag{7}$$

- Where

$$r(\hat{\beta}) = \frac{\left(\frac{1}{2} - \hat{\beta}\right)^2}{2\left(\frac{1}{2} - \hat{\beta}\right)^2 + \hat{\beta} + 1} \tag{8}$$

- And $\hat{\beta}$ is the regularity of the drift, and it means the Hurst parameter of fBm used is $H = 1 - \hat{\beta}$
- On the limit cases:
  - $\beta \to 0$, we have $r(\hat{\beta}) = 1/6 - \epsilon$
  - $\beta \to 1/2$, we have $r(\hat{\beta}) = \epsilon$

And about the implementation...

- One step towards the implementation comes from the fact that we want a mollified version of this derivative
- And we can compute $b \star p_{f_m} = \frac{d}{dx} B^H \star p_{f_m}$ as $B^H \star \frac{d}{dx} p_{f_m}$
- Some of the issues are:
  - The errors between consecutive Euler approximations don't seem to decrease as fast as they should
  - Implementing the drift can be challenging because of the vanishing variance

And about the implementation...

- One step towards the implementation comes from the fact that we want a mollified version of this derivative
- And we can compute $b \star p_{f_m} = \frac{d}{dx} B^H \star p_{f_m}$ as $B^H \star \frac{d}{dx} p_{f_m}$
- Some of the issues are:
  - The errors between consecutive Euler approximations don't seem to decrease as fast as they should
  - Implementing the drift can be challenging because of the vanishing variance

And about the implementation...

- One step towards the implementation comes from the fact that we want a mollified version of this derivative
- And we can compute $b \star p_{f_m} = \frac{d}{dx} B^H \star p_{f_m}$ as $B^H \star \frac{d}{dx} p_{f_m}$
- Some of the issues are:
  - The errors between consecutive Euler approximations don't seem to decrease as fast as they should
  - Implementing the drift can be challenging because of the vanishing variance

And about the implementation...

- One step towards the implementation comes from the fact that we want a mollified version of this derivative
- And we can compute $b \star p_{f_m} = \frac{d}{dx} B^H \star p_{f_m}$ as $B^H \star \frac{d}{dx} p_{f_m}$
- Some of the issues are:
  - The errors between consecutive Euler approximations don't seem to decrease as fast as they should
  - Implementing the drift can be challenging because of the vanishing variance

And about the implementation...

- One step towards the implementation comes from the fact that we want a mollified version of this derivative
- And we can compute $b \star p_{f_m} = \frac{d}{dx} B^H \star p_{f_m}$ as $B^H \star \frac{d}{dx} p_{f_m}$
- Some of the issues are:
  - The errors between consecutive Euler approximations don't seem to decrease as fast as they should
  - Implementing the drift can be challenging because of the vanishing variance

Conclusion

- Irregular coefficients on SDEs have gotten a lot of attention not only for theoretical reasons but because of their potential practical uses
- Theoretical results are useful but I believe it's necessary to give much more attention to the intrincacies of the implementations
- My implementation will potentially help generalise the numerical methods for one particular type of SDE

Conclusion

- Irregular coefficients on SDEs have gotten a lot of attention not only for theoretical reasons but because of their potential practical uses
- Theoretical results are useful but I believe it's necessary to give much more attention to the intrincacies of the implementations
- My implementation will potentially help generalise the numerical methods for one particular type of SDE

Conclusion

- Irregular coefficients on SDEs have gotten a lot of attention not only for theoretical reasons but because of their potential practical uses
- Theoretical results are useful but I believe it's necessary to give much more attention to the intrincacies of the implementations
- My implementation will potentially help generalise the numerical methods for one particular type of SDE

Buon appetito!

References I

Cannizzaro, Giuseppe and Khalil Chouk (May 2018). "Multidimensional SDEs with Singular Drift and Universal Construction of the Polymer Measure with White Noise Potential". In: *The Annals of Probability* 46.3.

Chaudru de Raynal, Paul-Eric and Stéphane Menozzi (Feb. 16, 2022). *On Multidimensional Stable-Driven Stochastic Differential Equations with Besov Drift*. arXiv: 1907.12263 [math]. URL: http://arxiv.org/abs/1907.12263 (visited on 06/18/2023). preprint.

Dareiotis, Konstantinos and Máté Gerencsér (Jan. 1, 2020). "On the Regularisation of the Noise for the Euler-Maruyama Scheme with Irregular Drift". In: *Electronic Journal of Probability* 25 (none). arXiv: 1812.04583.

References II

- De Angelis, Tiziano, Maximilien Germain, and Elena Issoglio (Sept. 13, 2022). "A Numerical Scheme for Stochastic Differential Equations with Distributional Drift". In: *Stochastic Processes and their Applications*.
- Delarue, François and Roland Diel (June 2016). "Rough Paths and 1d SDE with a Time Dependent Distributional Drift: Application to Polymers". In: *Probability Theory and Related Fields* 165.1-2.
- Goudenège, Ludovic, El Mehdi Haress, and Alexandre Richard (July 6, 2022). *Numerical Approximation of SDEs with Fractional Noise and Distributional Drift*. URL: https://hal-centralesupelec.archives-ouvertes.fr/hal-03715427 (visited on 09/05/2022). preprint.

References III

Gyongy, Istvan and Teresa Martinez (Dec. 1, 2001). "On Stochastic Differential Equations with Locally Unbounded Drift". In: *Czechoslovak Mathematical Journal* 51.4.

Higham, Desmond J. (Jan. 1, 2001). "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations". In: *SIAM Review* 43.3.

Issoglio, Elena and Francesco Russo (Aug. 23, 2022). *SDEs with Singular Coefficients: The Martingale Problem View and the Stochastic Dynamics View*. arXiv: 2208.10799 [math]. URL: http://arxiv.org/abs/2208.10799 (visited on 09/05/2022). preprint.

References IV

- Kloeden, Peter E. and Eckhard Platen (1999). *Numerical Solution of Stochastic Differential Equations*. Corr. 3rd print. Applications of Mathematics 23. Berlin ; New York: Springer. 636 pp.
- Kloeden, Peter E., Eckhard Platen, and Henri Schurz (1997). *Numerical Solution of SDE through Computer Experiments*. Corr. 2nd print. Universitext. Berlin ; New York: Springer. 292 pp.
- Talay, Denis (1990). "Simulation and Numerical Analysis of Stochastic Differential Systems : A Review". report. INRIA.
- Veretennikov, A. J. (Apr. 30, 1981). "On Strong Solutions and Explicit Formulas for Solutions of Stochastic Integral Equations". In: *Mathematics of the USSR-Sbornik* 39.3.